Information-theoretic methods in statistical machine learning

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Introduction

Era of massive data sets
Leads to new issues that are both statistical and computational in nature.
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1. Statistical issues
   - concentration of measure
   - curse of dimensionality
   - importance of “low-dimensional” structure
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   - importance of “low-dimensional” structure

2. Algorithmic issues
   - Increasing importance of privacy
   - Memory and storage constraints
   - Computational constraints
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1 Statistical issues
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2 Algorithmic issues
   ▶ Increasing importance of privacy
   ▶ Memory and storage constraints
   ▶ Computational constraints

This lecture
Some vignettes in which information theory has an important role to play.
Issue A: Privacy versus statistical utility

Many sources of data have both statistical utility and privacy concerns.

(a) Personal genome project
Issue A: Privacy versus statistical utility

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(a) Personal genome project

(b) Privacy breach
Scientific American, August 2013
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Think Before You Spit

Question

How to obtain principled tradeoffs between these competing criteria?
Issue B: Need for distributed estimators

Many modern datasets are too large to be stored on a single machine.

Google server farms

Netflix data base
**Issue B: Need for distributed estimators**

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*Google server farms*  

*Netflix data base*

**Question**

How to design statistical estimators that operate only on small pieces data? Fundamental tradeoffs between centralized and distributed estimators?
Issue C: Computational vs. statistical efficiency
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Question
What are the trade-offs between computational and statistical efficiency?
More specific questions

- How to reduce computational complexity while retaining statistical optimality?
- When is there a gap between polynomial-time and exponential-time algorithms?
- Differences in hierarchies of polynomial computation?
§1. Statistics and privacy

Privacy concerns with many types of data:

- your personal genome
- your sexual behaviour
- a company’s designs and algorithms
- your financial data
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Off-set by potential benefits from statistical aggregation:

- biological basis of disease
- epidemiological control
- reduced use of energy/materials
- improved economic forecasting
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Question

What are the fundamental trade-offs between privacy and statistical utility?
Basic model of local privacy

- each individual $i \in \{1, 2, \ldots, n\}$ has personal data $X_i \sim \mathbb{P}_{\theta^*}$
- conditional distribution $\mathbb{Q}$ between private data $X_1^n$ and public data $Z_1^n$
- estimator $Z_1^n \leftrightarrow \hat{\theta}$ of unknown parameter $\theta^*$.
Local privacy at level $\alpha$

**Definition**

Conditional distribution $Q$ is locally $\alpha$-differentially private if

$$e^{-\alpha} \leq \sup_z \frac{Q(z \mid x^n_1)}{Q(z \mid \bar{x}^n_1)} \leq e^{\alpha}$$

for all $x^n_1$ and $\bar{x}^n_1$ such that $d_{\text{HAM}}(x^n_1, \bar{x}^n_1) = 1$.

(Dwork et al., 2006)
Hypothesis testing interpretation

- consider two data sets $x_1^n$ and $\bar{x}_1^n$ that differ in at least one co-ordinate

- given privatized observations $Z_1^n$, an adversary wants to test between:
  - $H_0$: Underlying data set is $x_1^n = \{x_1, \ldots, x_n\}$
  - $H_1$: Underlying data set is $\bar{x}_1^n = \{\bar{x}_1, \ldots, \bar{x}_n\}$
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$\alpha$-privacy limits testing accuracy

For any test function $\psi : Z_1^n \rightarrow \{0, 1\}$:

$$\frac{1}{2} \sum_{j=0}^{1} \mathbb{P}[\psi(Z_1^n) \neq j, \ H = H_j] \geq \frac{1}{1 + e^\alpha}.$$ 

Consequently, $\alpha$-privacy provides a bound on the disclosure risk.

(Wasserman & Zhou, 2011)
Testing error versus privacy level

![Testing error versus privacy level graph](image)
Add $\alpha$-Laplacian noise

$$Z = x + W,$$

where $W$ has density $\propto e^{-\alpha |w|}$

(Dwork et al., 2006)
Add $\alpha$-Laplacian noise \cite{Dwork et al., 2006}

$$Z = x + W,$$

where $W$ has density $\propto e^{-\alpha |w|}$

For all $x, x' \in [-1/2, 1/2]$:

$$\sup_{z \in \mathbb{R}} \left| \log \frac{Q(z | x)}{Q(z | \bar{x})} \right| = \alpha \sup_{z \in \mathbb{R}} |z - x| - |z - \bar{x}| \leq \alpha.$$
Various mechanisms for $\alpha$-privacy

Choices from past work:

- randomized response in survey questions (Warner, 1965)
- Laplacian noise (Dwork et al., 2006)
- exponential mechanism (McSherry & Talwar, 2007)
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Some past work on privacy and estimation:

- local differential privacy and PAC learning (Kasiviswanathan et al., 2008)
- linear queries over discrete-valued data sets (Hardt & Rothblum, 2010)
- global differential privacy and histogram estimators (Hall et al., 2011)
- lower bounds for certain 1-D statistics (Chaudhuri & Hsu, 2012)
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**Question**

Can we provide a general characterization of the trade-offs between $\alpha$-privacy and statistical utility?

Will measure statistical utility in terms of minimax risk
Minimax optimality with $\alpha$-privacy

- family of distributions $\{\mathbb{P} \in \mathcal{F}\}$, and functional $\mathbb{P} \mapsto \theta(\mathbb{P})$
- samples $X_1^n \equiv \{X_1, \ldots, X_n\} \sim \mathbb{P}$ and estimator $X_1^n \mapsto \hat{\theta}(X_1^n)$
- loss function (e.g., squared error, 0-1 error, $\ell_1$-error)

$$(\hat{\theta}, \theta) \mapsto \mathcal{L}(\hat{\theta}, \theta)$$

quality of $\hat{\theta}$ as estimate of $\theta$
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Ordinary minimax risk:

$$
\mathcal{M}_n(\mathcal{F}) := \inf_{\hat{\theta}} \sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}\left[ \mathcal{L}(\hat{\theta}(X_1^n), \theta(\mathbb{P})) \right]
$$

Best estimator Worst-case distribution
Minimax optimality with $\alpha$-privacy

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$$\begin{align*}
(\hat{\theta}, \theta) & \mapsto L(\hat{\theta}, \theta) \\
\text{quality of } \hat{\theta} \text{ as estimate of } \theta
\end{align*}$$

Ordinary minimax risk:

$$M_n(\mathcal{F}) := \inf_{\hat{\theta}} \sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}[L(\hat{\theta}(X_1^n), \theta(\mathbb{P}))]$$

Best estimator Worst-case distribution

Minimax risk with $\alpha$-privacy

Estimators now depend on privatized samples $Z_1^n$

$$M_n(\alpha; \mathcal{F}) := \inf_{Q \in \mathcal{Q}_\alpha} \inf_{\hat{\theta}} \sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}[L(\hat{\theta}(Z_1^n), \theta(\mathbb{P}))]$$

Best $\alpha$-private channel
Suppose that we want to estimate the quantity $\mathbb{P} \mapsto \theta(\mathbb{P}) \equiv \text{density } f$
Illustration: Non-parametric density estimation

Suppose that we want to estimate the quantity $\mathbb{P} \mapsto \theta(\mathbb{P}) \equiv \text{density } f$

Classical fact

Ordinary minimax rates depend on number of derivatives $\beta > 1/2$ of density $f$:

$$\mathcal{M}_n(\mathcal{F}(\beta)) \asymp \left(\frac{1}{n}\right)^{\frac{2\beta}{2\beta+1}}$$  (Ibragimov & Hasminskii, 1978; Stone, 1980)
Optimal rates for $\alpha$-private density estimation

Consider density estimation based on $\alpha$-private views $(Z_1, \ldots, Z_n)$ of original samples $(X_1, \ldots, X_n)$. 
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**Theorem (Duchi, W. & Jordan, 2013)**

For all privacy levels $\alpha \in (0, 1/4]$ and smoothness levels $\beta > 1/2$:

$$M_n(\alpha; F(\beta)) \asymp \left(\frac{1}{\alpha^2 n}\right)^{\frac{2\beta}{2\beta+2}}$$
Optimal rates for \( \alpha \)-private density estimation

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* can give a simple/explicit scheme that achieves this optimal rate.*
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- contrast with classical rate $(1/n)^{\frac{2\beta}{2\beta + 1}}$: Penalty for privacy can be significant!
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- contrast with classical rate $(1/n)^{2\beta/(2\beta + 1)}$: Penalty for privacy can be significant!

**Example:** How many samples $N(\epsilon)$ to achieve error $\epsilon = 0.01$ for Lipschitz densities ($\beta = 1$)?

Classical case $N \approx 1,000$ versus Private case $N \approx 10,000$. 
How to measure “size” of function classes?
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- A $2\delta$-packing of $\mathcal{F}$ is a collection 
  $\{f^1, \ldots, f^M\} \subset \mathcal{F}$ such that
  $$\|f^j - f^k\|_2 > 2\delta \quad \text{for all } j \neq k.$$

- The packing number $M(2\delta)$ is the cardinality of the largest such set.
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Packing/covering entropy: introduced in seminal work by Kolmogorov

Deterministic concept but connected to Shannon entropy via volume ratios

Central object in proving minimax lower bounds (e.g., Hasminskii & Ibragimov, 1978; Birge, 1983; Yu, 1997; Yang & Barron, 1999)
From metric entropy to Fano’s inequality

- Construct a $2\delta$-packing of densities $\{f^1, \ldots, f^M\}$ with $\log M(\delta) \asymp (1/\delta)^{1/\beta}$ elements.
- Draw packing index $V \in \{1, \ldots M\}$ uniformly at random.
- Conditioned on $V = j$, draw $n$ i.i.d. samples $\{X_1, \ldots, X_n\} \sim f^j$. 
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### Mutual information

Relative discriminability controlled by mutual information

$$I(X_1, \ldots, X_n; V) = \frac{1}{M} \sum_{j=1}^{M} D(\bar{P} \parallel P^j)$$

where $\bar{P} = \frac{1}{M} \sum_{j=1}^{M} P^j$

Mixture distribution
A quantitative data processing inequality

- packing index $V \in \{1, 2, \ldots, M\}$
- non-private variables $(X \mid V = j) \sim \mathbb{P}_j$
- mixture distribution $\mathbb{P} = \frac{1}{M} \sum_{j=1}^{M} \mathbb{P}_j$. 

Diagram:

- $V$ → $X$ → $Z$
A quantitative data processing inequality

packing index $V \in \{1, 2, \ldots, M\}$

non-private variables $(X \mid V = j) \sim P_j$

mixture distribution $\overline{P} = \frac{1}{M} \sum_{j=1}^{M} P_j$.

---

Theorem (Duchi, W. & Jordan, 2013)

For any non-interactive $\alpha$-private channel $Q$, we have

$$\frac{I(Z_1, \ldots, Z_n; V)}{n} \leq (e^\alpha - 1)^2 \sup_{\|\gamma\|_\infty \leq 1} \left\{ \frac{1}{M} \sum_{j=1}^{M} \left( \int_{\mathcal{X}} \gamma(x) (dP_j(x) - d\overline{P}(x)) \right)^2 \right\}$$

dimension-dependent contraction
§2: Fast algorithms via randomized approximations

Massive data sets require very fast algorithms but with rigorous guarantees.
2: Fast algorithms via randomized approximations

Massive data sets require very fast algorithms but with rigorous guarantees.

**A general purpose tool:**

- Choose a random subspace of “low” dimension $m$.
- Project data into subspace, and solve reduced dimension problem.
§2: Fast algorithms via randomized approximations

A general purpose tool:

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Widely used in practice:

- Johnson & Lindenstrauss (1984): for Hilbert spaces
- various algorithms in theoretical computer science: (e.g., Vempala, 2004)
Randomized sketching of constrained least-squares

Original program based on data vector \( y \in \mathbb{R}^n \) and data matrix \( A \in \mathbb{R}^{n \times d} \):

\[
x^{\text{LS}} = \arg \min_{x \in \mathcal{C}} \left\{ \| A x - y \|_2^2 \right\}_{f(x)}
\]

where \( \mathcal{C} \) is a compact, convex set in \( \mathbb{R}^d \).
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$$x^{LS} = \arg \min_{x \in C} \|Ax - y\|_2^2$$

where $C$ is a compact, convex set in $\mathbb{R}^d$.

Given a sketching matrix $S \in \mathbb{R}^{m \times n}$, consider sketched version

$$\hat{x} = \arg \min_{x \in C} \|SAx - Sy\|_2^2$$
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**Question:**
How many projections $m$ for a required for a $\delta$-approximation?

Cost approx.: $f(x^{\text{LS}}) \leq f(\hat{x}) \leq (1 + \delta)^2 f(x^{\text{LS}})$.

Solution approx.: $\|\hat{x} - x^{\text{LS}}\|_A^2 \leq \delta^2 \|x^{\text{LS}}\|_A^2$. 

Optimal value: Sketch value: Approx. factor 

Sketch error: Approx. factor
Original problem based on data \((y, A) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}\):

\[
x^* = \arg \min_{x \in \mathbb{R}^d} \|Ax - y\|_2^2
\]
The classical sketch

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Given sketched data \((Sy, SA) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}\), compute sketched solution:

\[
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\[ Sy \quad SA \quad = \quad S \quad y \quad A \]

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Some past work:

- \(\delta\)-cost approximation using \(m \gtrsim \frac{1}{\delta^2} \text{rank}(A) \log(d)\) samples \((\text{Sarlos, 2006; Mahoney, 2011})\)
- various extensions and variants \((\text{Boutsidis & Drineas, 2009; Drineas et al., 2011})\)
- sharp results for general convex constraint sets \((\text{Pilanci & W., 2014})\)
Cost approximation: Unconstrained LS

Unconstrained Least Squares: $d = 500$

Approx. ratio $f(x)/f(x^*)$

Control parameter $\alpha$

Sketch size $m \gtrapprox \alpha d$
Solution approximation: Unconstrained LS

Mean–squared pred. error vs. row dimension

Sketch size $m \gtrapprox d \log(n)$
An information-theoretic lower bound

Consider random ensemble of least-squares problems

\[ x^{LS} = \arg \min_{x \in C} \left\{ \| y - Ax \|_2^2 \right\} \text{ where } y = Ax^* + w \]

with \( w \sim N(0, \sigma^2 I_n) \).
An information-theoretic lower bound

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**Theorem (Pilanci & W, 2014)**

For a broad class of random sketching matrices \( S \in \mathbb{R}^{m \times n} \), any estimator \( \tilde{x} \) based on the pair \((SA, Sy)\) has MSE lower bounded as

\[
\sup_{x^* \in C \cap B_2(1)} \mathbb{E}_{S,w} \left[ \| \tilde{x} - x^* \|_A^2 \right] \gtrsim \frac{\sigma^2 \log M}{\min\{m, n\}},
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where \( M \) is the \( 1/2 \)-packing number of \( \mathcal{C} \cap \mathbb{B}_2(1) \).

**Special case:** For unconstrained least-squares, this theorem implies that

\[
\sup_{x^* \in \mathbb{B}_2(1)} \mathbb{E}_{S,w} \left[ \| \tilde{x} - x^* \|_A^2 \right] \gtrsim \frac{\sigma^2 d}{\min\{m,n\}}.
\]
A different approach: Hessian sketch

Initial idea: Severe loss of information is caused by sketching data vector $y$. So let’s sketch only data matrix $A$, and solve program

$$\tilde{x} = \arg\min_{x \in \mathcal{C}} \left\{ \frac{1}{2m} \|SAx\|_2^2 - \langle A^T y, x \rangle \right\}.$$
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$$
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$$

Resulting bound **suffers from same problem** as classical sketch:

$$
\| \tilde{x} - \hat{x} \|^2_A \lesssim \delta^2 \| x^{LS} \|^2_A.
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Resulting bound **suffers from same problem** as classical sketch:

$$
\|\tilde{x} - \hat{x}\|_A^2 \precsim \delta^2 \|x_{LS}\|_A^2.
$$

....but this procedure can be iterated!
A different approach: Hessian sketch

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Resulting bound suffers from same problem as classical sketch:

\[
\|\tilde{x} - \hat{x}\|_A^2 \gtrsim \delta^2 \| x_{LS} \|_A^2.
\]

….but this procedure can be iterated!

**Iterative Hessian sketch**

Given an iteration number \( N \geq 1 \):

1. **(1)** Initialize at \( x^0 = 0 \).
2. **(2)** For iterations \( t = 0, 1, 2, \ldots, N - 1 \), generate an independent sketch matrix \( S^{t+1} \in \mathbb{R}^{m \times n} \), and perform the update

\[
x^{t+1} = \arg\min_{x \in \mathcal{C}} \left\{ \frac{1}{2m} \| S^{t+1} A(x - x^t) \|_2^2 - \langle A^T (y - Ax^t), x \rangle \right\}.
\]
3. **(3)** Return the estimate \( \hat{x} = x^N \).
Empirically: geometric convergence is observed

Error to sparse least-squares soln vs. iteration

Log error to sparse least-squares soln

Iteration number

\[\gamma = 2\]
\[\gamma = 5\]
\[\gamma = 25\]
Gaussian width of transformed tangent cone

$W(\mathcal{AK}) := \mathbb{E} \left[ \sup_{z \in \mathcal{AK} \cap S^{n-1}} \langle w, z \rangle \right] \text{ where } w \sim N(0, I_{n \times n}) \text{ and } S^{n-1} = \{z \in \mathbb{R}^n \mid \|z\|_2 = 1\}$
Main result for sub-Gaussian sketches

Tangent cone at $x^*$:

$$\mathcal{K} = \{ \Delta \in \mathbb{R}^d \mid \Delta = t(x - x^*) \in \mathcal{C} \text{ for some } t \geq 0 \}.$$ 

Width of transformed cone $AK \cap S^{n-1}$:

$$\mathcal{W}(AK) = \mathbb{E} \left[ \sup_{z \in AK \cap S^{n-1}} \langle w, z \rangle \right].$$

**Theorem (Pilanci & W., 2014)**

For any $\delta \in (0, 1)$, performing $\log(2/\delta)$ steps of iterative Hessian sketch using a sub-Gaussian sketch dimension lower bounded as

$$m \gtrsim \mathcal{W}^2(AK)$$

suffices to ensure that the sketched solution is $\delta$-optimal with probability at least $1 - c_1 e^{-c_2 m \delta^2}$. 
Sketching using randomized orthonormal systems

**Step 1:** Choose some fixed orthonormal matrix \( H \in \mathbb{R}^{n \times n} \).

Example: Hadamard matrices

\[
H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad H_{2^t} = \underbrace{H_2 \otimes H_2 \otimes \cdots \otimes H_2}_{\text{Kronecker product } t \text{ times}}
\]

(E.g., Ailon & Liberty, 2010)
Sketching using randomized orthonormal systems

Step 1: Choose some fixed orthonormal matrix $H \in \mathbb{R}^{n \times n}$. Example: Hadamard matrices

$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad H_{2^t} = H_2 \otimes H_2 \otimes \cdots \otimes H_2$$

Kronecker product $t$ times

$$S y = \tilde{H}$$

Step 2:

(A) Multiply data vector $y$ with a diagonal matrix of random signs $\{-1, +1\}$

(B) Choose $m$ rows of $H$ to form sub-sampled matrix $\tilde{H} \in \mathbb{R}^{m \times n}$

(C) Requires $O(n \log m)$ time to compute sketched vector $S y = \tilde{S} D y$.

(E.g., Ailon & Liberty, 2010)
Summary

Many challenges with massive data sets

- data collection and privacy concerns
- randomized algorithms for fast optimization
- distributed algorithms for statistical inference
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Some papers/pre-prints: